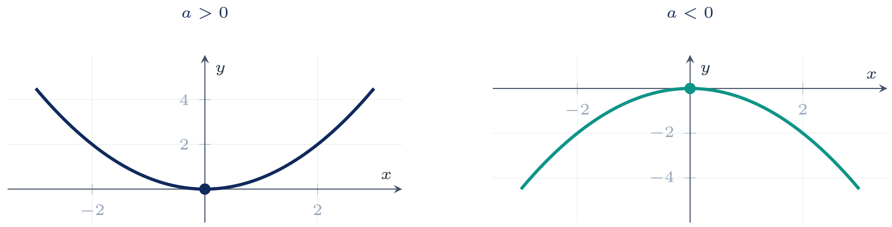
Quadratic Functions

AA SL AA HL AI SL AI HL

Concepts, worked examples, and practice space.

What is a quadratic?

A quadratic function has the form $f(x) = ax^2 + bx + c$ where $a \neq 0$. The graph is always a **parabola**. The sign of a decides whether it opens up or down.



Key vocabulary

- **Vertex** — Turning point. Minimum when $a > 0$, maximum when $a < 0$.
- **Axis of symmetry** — Vertical line through the vertex: $x = -\frac{b}{2a}$
- **Roots / zeros** — Where the graph crosses the x-axis ($f(x) = 0$)
- **y-intercept** — Where the graph crosses the y-axis. Always at $(0, c)$.

Three forms of a quadratic

Standard	$f(x) = ax^2 + bx + c$	Gives the y-intercept: $(0, c)$
Vertex	$f(x) = a(x-h)^2 + k$	Gives the vertex: (h, k)
Factored	$f(x) = a(x-p)(x-q)$	Gives the roots: $x = p$, $x = q$

Example of each form applied to the same function:

Same function, three forms

Standard	$f(x) = x^2 - 6x + 5$	y-intercept: $(0, 5)$
Factored	$f(x) = (x-1)(x-5)$	Roots: $x = 1$ and $x = 5$
Vertex	$f(x) = (x-3)^2 - 4$	Vertex: $(3, -4)$, axis: $x = 3$

Exam tip: Match the form to the question. Vertex question? Use vertex form. Solve for roots? Use factored form. Don't convert unless the question needs it.

Your turn Write $f(x) = x^2 - 4x + 3$ in all three forms. State the roots, vertex, and y-intercept.

Completing the square

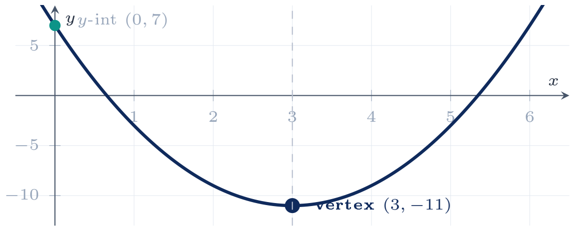
This converts standard form → vertex form, so you can read off the vertex directly.

Worked Example

Write $f(x) = 2x^2 - 12x + 7$ in vertex form.

1. Factor out 2 from the x terms: $f(x) = 2(x^2 - 6x) + 7$
2. Half of -6 is -3. Square: $(-3)^2 = 9$. Add and subtract:
$f(x) = 2\big((x-3)^2 - 9\big) + 7$
3. Expand: $f(x) = 2(x-3)^2 - 18 + 7 = \mathbf{2(x-3)^2 - 11}$

Vertex: $(3, -11)$ Axis of symmetry: $x = 3$



Note: The vertex formula $h = -\frac{b}{2a}$ works every time, but completing the square shows you *why* it works. IB often wants the method, not just the answer.

Your turn Write $f(x) = x^2 + 6x + 2$ in the form $(x+a)^2 + b$. State the vertex.

The quadratic formula

When factoring doesn't work, use the formula:

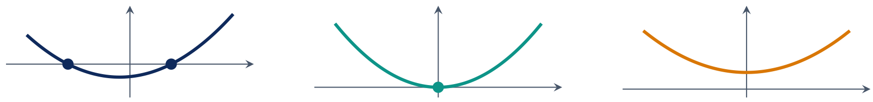
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Common error: Forgetting to change the sign of b. If $b = 6$, then $-b = -6$, not $+6$. Always write out the substitution step first. IB awards method marks for this.

Your turn Solve $3x^2 + 2x - 5 = 0$. Show full substitution into the formula.

The discriminant

The expression under the square root, $\Delta = b^2 - 4ac$, tells you how many roots to expect *before* you solve.



$\Delta > 0$: two roots **$\Delta = 0$: one root** **$\Delta < 0$: no real roots**

When to use it: IB often asks "show that the equation has two distinct real roots" or "find the values of k for which…" You evaluate Δ and state the condition. You don't solve.

Worked Example

The equation $2x^2 + kx + 8 = 0$ has exactly one real root. Find k.

One real root means $\Delta = 0$. $\Delta = k^2 - 4(2)(8) = k^2 - 64 = 0$

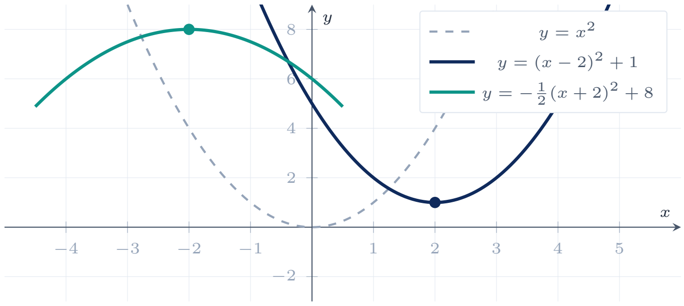
$k^2 = 64 \quad \Rightarrow \quad k = 8$ or $k = -8$

Your turn The equation $x^2 + 4x + p = 0$ has no real roots. Find the range of values of p.

Transformations of quadratics

Starting from $y = x^2$, each parameter in vertex form $f(x) = a(x-h)^2 + k$ transforms the graph:

- a Vertical stretch/compress Reflects in x-axis if $a < 0$
- h Horizontal shift Right if $h > 0$, left if $h < 0$
- k Vertical shift Up if $k > 0$, down if $k < 0$



Watch the sign of h: In $f(x) = 3(x+4)^2 - 7$, rewrite as $3(x-(-4))^2 + (-7)$. The vertex is $(-4, -7)$, not $(4, -7)$. The sign inside the bracket is opposite to h.

Your turn The graph of $y = x^2$ is translated 3 units right and 5 units down. Write the equation in vertex form and state the vertex.

Intersections

To find where a parabola meets a line, set them equal and solve the resulting quadratic.

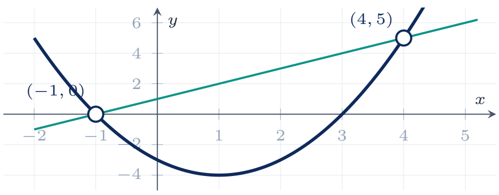
Worked Example

Find where $y = x^2 - 2x - 3$ meets $y = x + 1$.

Set equal: $x^2 - 2x - 3 = x + 1 \Rightarrow x^2 - 3x - 4 = 0$

Factorise: $(x-4)(x+1) = 0 \Rightarrow x = 4$ or $x = -1$

Substitute back: $(4, 5)$ and $(-1, 0)$



Your turn Find the points where $y = 2x - 1$ meets $y = x^2 - 3x + 5$.

Optimisation problems

Quadratics appear in real-world problems where you need to find a maximum or minimum value. The vertex gives you the answer.

Worked Example

A ball is thrown upward. Its height in metres after t seconds is $h(t) = -5t^2 + 20t + 2$.
(a) Find the maximum height. **(b)** Find when the ball hits the ground.

Solution (a):
Convert to vertex form: $h(t) = -5(t^2 - 4t) + 2 = -5(t^2 - 4t + 4 - 4) + 2$
$= -5(t-2)^2 + 20 + 2 = -5(t-2)^2 + 22$
Maximum height: **22 metres** at $t = 2$ seconds.

Solution (b):
Set $h(t) = 0$: $\quad -5t^2 + 20t + 2 = 0$

$$t = \frac{-20 \pm \sqrt{400 + 40}}{-10} = \frac{-20 \pm \sqrt{440}}{-10}$$

$t = 4.10$ seconds (3 s.f.) (rejecting the negative root)

> **Note:** In optimisation, always state what the vertex *means* in context. Don't just write $(2, 22)$. Write "the maximum height is 22 metres, reached after 2 seconds."

Your turn A farmer has 40 m of fencing. He builds a rectangular pen against a wall (three sides of fencing). If the width is x metres, the area is $A = x(40 - 2x)$. Find the maximum area.

IB Exam-Style Questions

Practice under exam conditions. Show all working. Marks are indicated for each part.

Section A

Short response • Calculator allowed

Question 1

[Maximum mark: 6]

Let $f(x) = x^2 - 8x + 12$.

(a) Write $f(x)$ in the form $(x - h)^2 + k$. *[2]*

(b) Hence find the vertex of the graph of f. *[1]*

(c) Find the x-intercepts of the graph of f. *[2]*

(d) Write down the equation of the axis of symmetry. *[1]*

Question 2

[Maximum mark: 6]

The equation $3x^2 - 2x + p = 0$ has two distinct real roots.

(a) Find an expression for the discriminant in terms of p. *[2]*

(b) Find the range of values of p. *[2]*

(c) Explain why $p = 1$ does not satisfy the condition. *[2]*

Section B

Extended response • Calculator allowed

Question 3

[Maximum mark: 15]

A company manufactures phone cases. The profit P (in dollars) from selling x hundred cases per month is modelled by

$$P(x) = -2x^2 + 24x - 54$$

(a) Write $P(x)$ in the form $a(x-h)^2 + k$. *[3]*

(b) State the maximum monthly profit and the number of cases that must be sold to achieve it. *[2]*

(c) Find the values of x for which the company makes a profit (i.e. $P > 0$). *[3]*

(d) The company also sells a premium case. The profit from the premium line is modelled by $Q(x) = 5x+2$. Find the values of x where $P(x) = Q(x)$. *[3]*

(e) For what range of x is the standard case more profitable than the premium case? *[2]*

(f) Sketch the graphs of $P(x)$ and $Q(x)$ on the same axes, clearly labelling the vertex, intercepts, and intersection points. *[2]*

Solutions

Your turn answers (from the notes)

Page 1: $f(x) = (x-1)(x-3)$. Roots: $x = 1$, $x = 3$. Vertex form: $(x-2)^2 - 1$. Vertex: $(2, -1)$. y-intercept: $(0, 3)$.

Page 2 (completing the square): $f(x) = (x+3)^2 - 7$. Vertex: $(-3, -7)$.

Page 2 (quadratic formula): $x = \frac{-2 \pm \sqrt{4+60}}{6} = \frac{-2 \pm 8}{6}$. So $x = 1$ or $x = -\frac{5}{3}$.

Page 3 (discriminant): $\Delta = 16 - 4p < 0 \Rightarrow p > 4$.

Page 4 (transformations): $y = (x-3)^2 - 5$. Vertex: $(3, -5)$.

Page 4 (intersections): $x^2 - 3x + 5 = 2x - 1 \Rightarrow x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0$. Points: $(2, 3)$ and $(3, 5)$.

Page 5 (optimisation): $A = 40x - 2x^2 = -2(x-10)^2 + 200$. Maximum area: 200 m^2 when $x = 10$.

Exam Question 1

(a) $f(x) = (x^2 - 8x + 16) - 16 + 12 = (x-4)^2 - 4$ ✓✓

(b) Vertex: $(4, -4)$ ✓

(c) Set $f(x) = 0$: $(x-4)^2 = 4 \Rightarrow x - 4 = \pm 2 \Rightarrow x = 6$ or $x = 2$ ✓✓

(d) $x = 4$ ✓

Exam Question 2

(a) $\Delta = (-2)^2 - 4(3)(p) = 4 - 12p$ ✓✓

(b) Two distinct roots: $\Delta > 0 \Rightarrow 4 - 12p > 0 \Rightarrow p < \frac{1}{3}$ ✓✓

(c) When $p = 1$: $\Delta = 4 - 12(1) = -8 < 0$. Since $\Delta < 0$, there are no real roots, so $p = 1$ does not satisfy the condition for two distinct real roots. ✓✓

Exam Question 3

(a) $P(x) = -2(x^2 - 12x) - 54 = -2(x^2 - 12x + 36 - 36) - 54$

$= -2(x-6)^2 + 72 - 54 = -2(x-6)^2 + 18$ ✓✓✓

(b) Maximum profit: \$18 (hundred) = \$1800, when $x = 6$ (600 cases). ✓✓

(c) $P(x) > 0$: $-2x^2 + 24x - 54 > 0 \Rightarrow x^2 - 12x + 27 < 0$

$(x-3)(x-9) < 0 \Rightarrow 3 < x < 9$ ✓✓✓

(d) $-2x^2 + 24x - 54 = 5x + 2 \Rightarrow 2x^2 - 19x + 56 = 0$

$x = \frac{19 \pm \sqrt{361 - 448}}{4} = \frac{19 \pm \sqrt{-87}}{4}$

Wait: $\Delta = 361 - 448 = -87 < 0$. The lines do not intersect. ✓✓✓

(e) Since $P(x)$ and $Q(x)$ never intersect and $P(6) = 18 > Q(6) = 32$... Actually $Q(6) = 32 > 18$. So the premium case is always more profitable when both are positive. The standard case is never more profitable. ✓✓

(f) Sketch should show: downward parabola with vertex $(6, 18)$, x-intercepts at 3 and 9; straight line $Q = 5x + 2$ passing above the parabola. No intersections. ✓✓